# Curves on Complete Intersections and Measures of Irrationality

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**Q.** $X \hookrightarrow \mathbb{P}^N$ if $\Lambda$ linear space of dim $n+1$
dim $= n$
then $X$ is covered by curves of the form $X \cap \Lambda$.

when do there exist "simpler" curves than linear slices?

**Thm A:** if $X \hookrightarrow \mathbb{P}^{n+r}$ is a general complete intersection of dim $n$ and codim $r$ cut out by equations of degree $(d_1, \ldots, d_r)$ then for $d_1, \ldots, d_r \geq N(n,r)$ we have

$$\deg C \geq d_1 \cdots d_r \quad \text{for any } C \subseteq X.$$

moreover if $d_1, \ldots, d_r \geq 2n$ then

$$\deg C \geq (d_1 - 2n+1) \cdots (d_r - 2n+1)$$

## § Measures of Irrationality:

$$\mathrm{irr}(X) := \min\{\deg f \mid X \overset{f}{\dashrightarrow} \mathbb{P}^n \text{ dominant} \quad n = \dim X\}$$

$$\mathrm{covgon}(X) := \min\{c \mid X \text{ is covered by curves of gonality } \leq c\}$$

$\mathrm{irr}(X) \geq \mathrm{covgon}(X)$.

**BDELU**: for $X \subseteq \mathbb{P}^{n+1}$ general hypersurface of degree $d$.

$$\mathrm{irr}(X) \geq \mathrm{covgon}(X) \geq d-n+1.$$

what about complete intersections?

**Thm B**: if $X \subseteq \mathbb{P}^{n+r}$ general complete intersection of type $(d_1, \dots, d_r)$ then $\forall \epsilon > 0$ there is $N(\epsilon, n, r)$ s.t. if $d_1, \dots, d_r \geq N$ then

$$\mathrm{covgon}(X) \geq (1-\epsilon)\, d_1 \cdots d_r$$

# § Thm A $\Rightarrow$ Thm B :

Suppose $X$ is a smooth projective variety,

Consider

$$\begin{tikzcd} \mathbb{P}^1\times S \arrow[dr] & \mathcal{C} \arrow[l, "\deg = c"'] \arrow[r, "\text{dominant}", "f"'] \arrow[d] & X \\ & S & \end{tikzcd}$$

$S \leftarrow$ $\dim = n-1$.

Suppose I have $\omega_1, \dots, \omega_r \in H^0(X, \omega_X)$

$f^*\omega_1, \dots, f^*\omega_r$ top forms on $\mathcal{C}$.

$\mathrm{tr}\, f^*\omega_1, \dots, \mathrm{tr}\, f^*\omega_r$ top forms on $\mathbb{P}^1\times S$

hence zero.


$\eta \longmapsto (\mathrm{tr}\,\eta)$

$$(\mathrm{tr}\,\eta)_p = \sum_{x\in f^{-1}(p)} df^{-1}(\eta_x)$$

if $\omega_1, \dots, \omega_r$ separate many points

then I can ensure the sum is nonzero.

if $\omega_1, \dots, \omega_r$ separate $c$ general points of

$X$ then I can ensure that some trace is nonzero.

If $H^0(X, \omega_X)$ separates $c$ points of $X$ then $\operatorname{covgon}(X) \geq c+1$.

§ Separation of Points:

write $X \subseteq Y$ $\quad$ $Y$ is a complete intersection of type $(d_2, \ldots, d_r)$

$X$ is a divisor $\in |d_1 H|$.

Now $\quad K_X = (\underbrace{K_Y + d_1 H}_{})|_X$

To show separation of points it suffices to show

$$H^1(Y, \mathcal{O}(K_Y + d_1 H) \otimes I_{p_1, \ldots, p_c}) = 0 \quad \text{because then}$$

$$H^0(Y, \mathcal{O}(K_Y + d_1 H)) \to H^0(Y, \mathcal{O}(K_Y + d_1 H) \otimes \mathcal{O}_{p_1, \ldots, p_c})$$

is then surjective.

Strategy 1:

$$H^1(Y, \mathcal{O}(K_Y + d_1 H) \otimes I_{\{p_1, \ldots, p_c\}})$$

$= H^1(Bl_{p_1,\dots,p_c} Y, \mathcal{O}(K_{Bl\,Y} + \underbrace{d_1 H - n(E_1 + \dots + E_c)}_{\text{big + nef.}}))$

Strategy 2:

Suppose we had an ideal sheaf $J$ of $Z \subseteq X$ s.t. $p_1, \dots, p_c \in Z$ and $p_1$ is an isolated point and $H^1(Y, \mathcal{O}(K_Y + d_1 H) \otimes J) = 0$.

$$H^0(Y, \mathcal{O}(K_Y + d_1 H)) \twoheadrightarrow H^0(Z, \mathcal{O}_Z(K_Y + d_1 H))$$

get a section $s_1$ vanishes at $p_2, \dots, p_c$ and not at $p_1$.

get $s_2$ vanishes at $p_3, \dots, p_c$ and not at $p_2$.

$\begin{pmatrix} s_1 \\ \vdots \\ s_c \end{pmatrix}$ s.t. evaluations at $(p_1, \dots, p_c)$ is an upper $\Delta$ matrix $\Longrightarrow$ Separation of points.

Lemma: if $X$ is an $n$-dimensional variety and $p_1, \dots, p_c$ smooth points, $H$ ample $\mathbb{Q}$-div s.t. $H^n > c \cdot \gamma^n$ for some $\gamma > 0$ then

$\exists$ effective $\mathbb{Q}$-divisor

$$D \in |H|_{\mathbb{Q}} \quad \text{with} \quad \mathrm{mult}_{p_i}(D) > \gamma.$$

<u>Lemma</u>: if $X$ normal proj. variety of $\dim = n$ and $p \in X$ smooth point on $X$. If $D$ is an eff. $\mathbb{Q}$-div s.t. $\mathrm{mult}_p(D) \geq n$ then

$$p \in V(\mathcal{J}(X, D))$$

<u>Prop</u>: if $X$ is a normal $\mathbb{Q}$-gorenstein proj. variety of $\dim = n$, $p_1, \ldots, p_c \in X$ distinct smooth points $D$ an effective $\mathbb{Q}$-div s.t. $V(\mathcal{J}(X,D)) \ni p_1, \ldots, p_c$

for any ample $H$ and $\epsilon > 0$ $\exists$ rational $0 < r < \epsilon$ and eff. $\mathbb{Q}$-div $E \sim_{\mathbb{Q}} rH$ s.t.

(1). $V(\mathcal{J}(X, D+E))$ is irred. at $p_1$

(2) $\dim_{p_i} V(\mathcal{J}(X, D+E)) \leq \dim_{p_i} V(\mathcal{J}(X, D))$

for all $p_i$.

## § Proof of Thm A:

idea is degenerate $\underset{(d_1,\dots,d_r)}{X} \rightsquigarrow X_1 \cup_Z X_2$

$X_1$ · type $(a, d_2, \dots, d_r)$ $\quad a+b = d_1$

$X_2$ type $(b, d_2, \dots, d_r)$

$Z$ type $(a, b, d_2, \dots, d_r)$

Want: $C \subseteq X \rightsquigarrow \overline{C} \subset X_1 \cup_Z X_2$

$$\overline{C} = C_1 \cup C_2$$

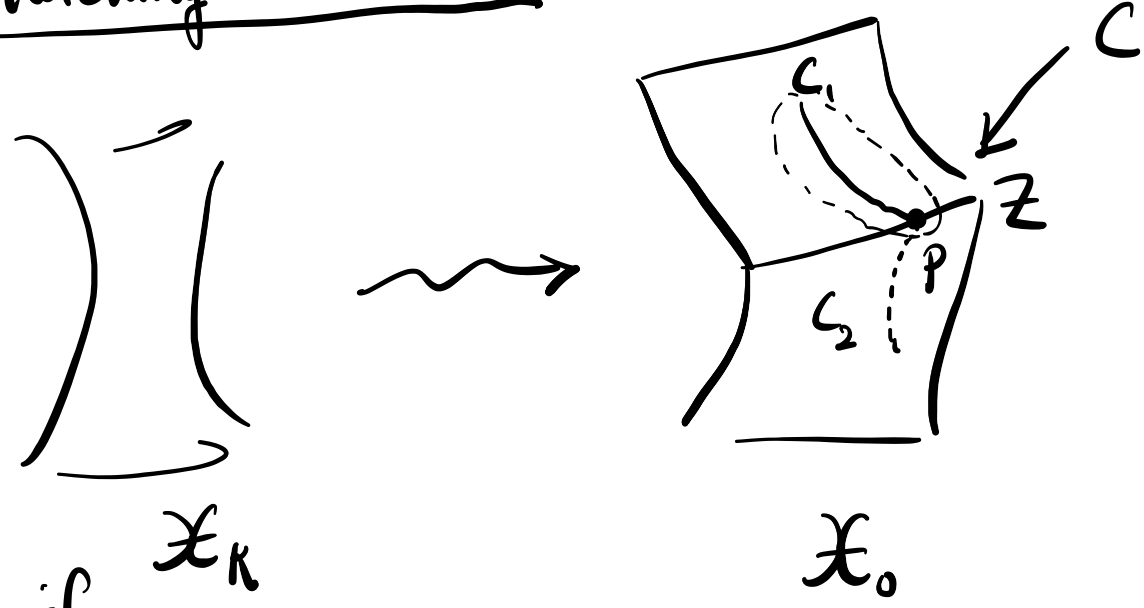
WRONG:

idea 1 $\mathcal{X}$ total family of degeneration

$\downarrow$ is singular at exactly those

$\mathbb{A}^1$ two points.

<u>Matching condition:</u>




if $C$ deforms to $\mathcal{X}_K$ then either

1. $p \in \mathcal{X}^{sing}$
2. $p \in C_2$ meeting $Z$ with same multiplicity as $C_1$
3. $p \in C_Z \subseteq Z$

<u>idea 2</u>: study covering family of curves.

if
$$\begin{array}{ccc} \mathcal{C} & \to & X \\ \downarrow & & \\ S & & \end{array}$$
dominant then specializing

to
$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & X_1 \cup_Z X_2 \\ \downarrow & & \end{array}$$
this is also dominant.

hence some curve hits

S                    $Z$ outside of $\mathcal{X}^{sing}$.

$$\mathrm{Corvdeg}(X) \geq \min\left\{\begin{array}{l}\mathrm{Corvdeg}(X_1) + \mathrm{corvdeg}(X_2), \\ \mathrm{Corvdeg}(Z)\end{array}\right\}$$

<u>Thm</u> C: $\mathrm{corvdeg}(X_{d_1,\dots,d_r}) \geq (d_1-n+1)\cdots(d_r-n+1)$

for $X_{d_1,\dots,d_r} \subseteq \mathbb{P}^{n+r}$ general.

Results of RY22 grassmanian teqchniques

Thm C $\Rightarrow$ Thm A.